Int. J. Heat Mass Transfer. Vol. 27, No. 1, pp. 69–75, 1984
Printed in Great Britain

0017-9310/84 $3.00 + 0.00

VERTICAL NATURAL CONVECTION FLOWS IN POROUS MEDIA: CALCULATIONS OF IMPROVED ACCURACY

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(*Received* 29 *September* 1982 *and in revised form* 26 *April* 1983)

Abstract—Higher-order corrections to the boundary layer solutions have been obtained for vertical natural convection flows in porous media. The analysis of Cheng and Chang [*Lett. Heat Mass Transfer* **6**, 253–258 (1979)] has been extended to a uniform heat flux surface condition and to plane plume flows. Matched asymptotic solutions up to $O(\varepsilon^2)$ have been obtained for these downstream temperature variations. For the isothermal surface condition, there was no correction in either temperature or velocity up to $O(\varepsilon)$. However, it has been found that the first eigenfunction for this condition coincides with the $O(\varepsilon^2)$ term in the inner expansion. This makes the second-order correction indeterminate. For the uniform surface flux flow, the ratio of the corrected local Nusselt number to its value from the simplest boundary layer result is

$$1+0.3333\varepsilon+0.0201\varepsilon^2.$$

For the plane plume, the ratio of the downstream centerline temperature excess to its value for the simplest solution is

$$1+0.4714\varepsilon+0.4760\varepsilon^2.$$

The Prandtl number is absorbed in the transformations. The new results are more accurate values of transport quantities for these flows, in the Rayleigh number range of practical interest.

NOMENCLATURE

A_1	constant in equation (34)
c_p	specific heat of the fluid
$d(x)$	downstream temperature decay for the zero-order boundary layer, Nx^n
f	nondimensional stream function, $\psi/(\alpha\, Ra_x^{1/2})$
g	acceleration due to gravity
h.o.t.	higher-order terms
h_x	local heat transfer coefficient
K	permeability of the porous medium
k	thermal conductivity of the saturated porous medium
N	constant in the equation for $d(x)$
Nu_x	local Nusselt number, $q''x/(t_o-t_\infty)k$
$(Nu_x)_{b.l.}$	local Nusselt number obtained from the zero-order solution
n	exponent in the equation for $d(x)$
q''	constant wall heat flux in equation (5), $-k(\partial t/\partial y)\vert_{y=0}$
Q	energy input at $x=0$ for the line plume, $2\int_0^\infty \rho u c_p(t-t_\infty)\,dy$
r	cylindrical radial coordinate, $(x^2+y^2)^{1/2}$
Ra_x	local Rayleigh number, $\rho g\beta x K d(x)/\mu\alpha$
t	temperature
u	velocity component in x-direction
v	velocity component in y-direction
x	vertical coordinate
y	horizontal coordinate.

Greek symbols

α	thermal diffusivity of the porous medium, $k/\rho c_p$
α_k	kth eigenvalue
β	coefficient of thermal expansion of fluid
δ	characteristic thickness of boundary layer, $x/(Ra_x)^{1/2}$
ε	perturbation parameter, $Ra_x^{-1/2}$
μ	dynamic viscosity of fluid
ρ	density of fluid
λ	$[\rho g\beta KN/\mu\alpha]^{-1/(n+1)}$
η	nondimensional horizontal coordinate, y/δ
ψ	stream function
θ	angular coordinate measured from the $+x$-axis.

Subscripts

o	condition at $y=0$
0	condition when $\varepsilon=0$
1	first-order correction
2	second-order correction
∞	value in the ambient
r	reference value.

INTRODUCTION

ANALYSIS of natural convection flow arising from a heated impermeable surface, embedded in fluid saturated porous media, has been used to model the heating of groundwater in an aquafier by a dike, which is idealized as a vertical impermeable surface [1]. Boundary layer formulation of Darcy's law and the energy equation were used. Subsequent studies of boundary layer natural and mixed convection flows are described in detail by Cheng [2].

To extend the range of applicability of the boundary layer analysis to relatively lower values of

Rayleigh number (i.e. relatively shorter downstream distances), Cheng and Chang [3] make a singular perturbation analysis, considering the first-order boundary layer corrections both for vertical and horizontal natural convection flows. For vertical flows, results are given for a power law type of surface temperature dependence, $t_o(x)$, downstream, in x. That is $(t_o - t_\infty) = Nx^n = d(x)$. The results are valid, then, only for a surface temperature variation specified *a priori*, such as isothermal. The uniform flux surface and the line plume cannot be treated by this analysis, since the surface temperature variation is improved successively at each order and is not imposed externally.

The present investigation considers three vertical natural convection flows in saturated porous media. The three types of motion-generating heating conditions are an isothermal surface, a constant flux surface and a plane plume, generated from a line heat source. Consistent higher-order approximations, including second order, have been developed, using the method of matched asymptotic expansions [4]. The approach used is similar to that of Hieber [5] and Mahajan and Gebhart [6] for Newtonian flows. Specifically, in this study it is shown that the boundary conditions previously applied in the higher-order analysis of the constant heat flux surface condition are not the appropriate ones.

ANALYSIS

The governing equations for two-dimensional (2-D) natural convection flow employing the Boussinesq approximations, in a fluid saturated porous medium are:

continuity equation

$$u = \psi_y, \quad v = -\psi_x, \tag{1}$$

Darcy's law

$$\psi_{yy} + \psi_{xx} = \frac{K}{\mu}\rho_r g\beta \frac{\partial}{\partial y}(t - t_\infty), \tag{2}$$

energy equation

$$\psi_y \frac{\partial t}{\partial x} - \psi_x \frac{\partial t}{\partial y} = \alpha\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right). \tag{3}$$

The applicable boundary conditions on t, u, and v, for each of the three flows considered, are:

(a) the isothermal surface

$$t = t_o, \quad v = 0 \quad \text{at} \quad y = 0, \quad x > 0, \tag{4a}$$

$$t \to t_\infty, \quad u \to 0 \quad \text{as} \quad r \to \infty, \quad \theta \neq 0, \tag{4b}$$

$$\partial u/\partial y = 0 = v = \partial t/\partial y \quad \text{at} \quad y = 0, \quad x < 0; \tag{4c}$$

(b) the uniform flux surface

$$q'' = -k\frac{\partial t}{\partial y}, \quad v = 0 \quad \text{at} \quad y = 0, \quad x > 0, \tag{5a}$$

$$t \to t_\infty, \quad u \to 0 \quad \text{as} \quad r \to \infty, \quad \theta \neq 0, \tag{5b}$$

$$\partial u/\partial y = 0 = v = \partial t/\partial y \quad \text{at} \quad y = 0, \quad x < 0; \tag{5c}$$

(c) the line source plume

$$v = \frac{\partial u}{\partial y} = \frac{\partial t}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad x > 0, \tag{6a}$$

$$t \to t_\infty, \quad u \to 0 \quad \text{as} \quad r \to \infty, \quad \theta \neq 0, \tag{6b}$$

$$\frac{\partial u}{\partial y} = v = \frac{\partial t}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad x < 0. \tag{6c}$$

The appropriate expansions for the inner region, corresponding to $y = O(\delta)$ and $x \geqslant O(\lambda)$, are given below. Here, δ is the characteristic boundary region thickness and λ is the extent of the leading edge region. The corrections due to the eigenfunctions are considered following the analysis. The stream and temperature functions are postulated as

$$\psi(x, y) = \alpha\, Ra_x^{1/2}[f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \text{h.o.t.}], \tag{7}$$

$$(t - t_\infty) = d(x)[\phi_0(\eta) + \varepsilon\phi_1(\eta) + \varepsilon^2\phi_2(\eta) + \text{h.o.t.}], \tag{8}$$

where $\varepsilon = Ra_x^{-1/2}$ and Ra_x is the local Rayleigh number defined by

$$Ra_x = \frac{\rho g\beta x K d(x)}{\mu\alpha}. \tag{9}$$

In equation (9), $d(x)$ is the assigned downstream temperature variation at $y = 0$, at the surface, for the zero-order equations. These equations are obtained by letting $\varepsilon = 0$ in equations (7) and (8). Expressed in another way, $d(x) \equiv (t_o - t_\infty)_0 = Nx^n$, the subscript '0' implying the zero-order value. In ref. [3], in the definition of the Rayleigh number, $d(x)$ is taken as $(t_o - t_\infty)_0 = Nx^n$. In general, $(t_o - t_\infty) \neq (t_o - t_\infty)_0$, that is, the downstream temperature distribution and level is indeed affected by the higher-order effects, or corrections, in equations (7) and (8). The $n = 1/3$ condition in ref. [3], then, does not result in a uniform heat flux surface condition. Additional x dependence in the surface heat flux is introduced by the higher-order corrections.

In the outer irrotational and isothermal region, corresponding to $\theta \neq 0$, and $r > O(\lambda)$, the following expansions are postulated

$$\psi = \tilde{\psi}_0 + \tilde{\psi}_1 + \tilde{\psi}_2 + , \ldots, \tag{10}$$

$$t = t_\infty. \tag{11}$$

The f_0 and ϕ_0 in equations (7) and (8) are the boundary layer solution in ref. [2], as follows:

$$f_0' - \phi_0 = 0, \tag{12}$$

$$\phi_0'' + \frac{(n+1)}{2} f_0\phi_0' - nf_0'\phi_0 = 0. \tag{13}$$

The zero-order boundary conditions are:

for (a)

$$n = 0; \quad f_0(0) = \phi_0(0) - 1 = \phi_0(\infty) = 0, \tag{14}$$

for (b)

$$n = \tfrac{1}{3};\quad f_0(0) = 1-\phi_0(0) = \phi_0(\infty) = 0, \tag{15}$$

for (c)

$$n = -\tfrac{1}{3};\quad f_0(0) = \phi_0'(0) = 1-\phi_0(0) = 0. \tag{16}$$

Numerical integration of equations (12) and (13), with the appropriate boundary conditions, yields:

for (a)

$$\phi_0'(0) = 0.444, \tag{17}$$

for (b)

$$\phi_0'(0) = -0.6777, \tag{18}$$

for (c), i.e. the plane plume, closed form solutions have been given by Yih [7]. In the present variables, these are

$$f_0 = \sqrt{6}\tanh[\eta/\sqrt{6}] \tag{19}$$

$$\phi_0 = \mathrm{sech}^2[\eta/\sqrt{6}]. \tag{20}$$

The function $\tilde{\psi}_0$ is governed by

$$\nabla^2\tilde{\psi}_0 = 0, \tag{21}$$

with

$$\tilde{\psi}_0(\theta = 0) = 0 = \tilde{\psi}_0(\theta = \pi), \tag{22}$$

which results in $\tilde{\psi}_0 \equiv 0$.

For $\tilde{\psi}_1$

$$\nabla^2\tilde{\psi}_1 = 0, \tag{23}$$

with

$$\tilde{\psi}_1(\theta = \pi) = 0,\quad \tilde{\psi}_1(\theta = 0) = \alpha f_0(\infty)\left(\frac{r}{\lambda}\right)^{(n+1)/2}. \tag{24}$$

The inhomogeneous condition in equation (24) results from matching of v from the boundary layer solution with the outer solution. The solution of equations (23) and (24) is

$$\tilde{\psi}_1 = \left(\frac{r}{\lambda}\right)^{(n+1)/2} f_0(\infty)\frac{\sin\{[(n+1)/2](\pi-\theta)\}}{\sin\{[(n+1)/2]\pi\}}. \tag{25}$$

This result is expanded around $\theta = 0$, noting that

$$\theta = \varepsilon\eta + \frac{\varepsilon^3}{3}\eta^3 + \text{h.o.t.},$$

and

$$r = x\left[1 + \frac{\varepsilon^2}{2}\eta^2 + \text{h.o.t.}\right].$$

In the matching region

$$\tilde{\psi}_1 = \alpha Ra_x^{1/2} f_0(\infty)\left[1 - \frac{(n+1)}{2}\cot\left\{(n+1)\frac{\pi}{2}\right\}\varepsilon\eta + \frac{(1-n^2)}{8}\varepsilon^2\eta^2 + \text{h.o.t.}\right]. \tag{26}$$

The first-order inner functions are next obtained as

$$f_1'' - \phi_1' = 0, \tag{27}$$

and

$$\phi_1'' - \frac{(n-1)}{2}f_0'\phi_1 - nf_1'\phi_0 + \frac{(n+1)}{2}f_0\phi_1' = 0. \tag{28}$$

The boundary conditions for the first correction for each flow are:

(a)

$$f_1(0) = \phi_1(0) = \phi_1(\infty) = 0,\quad f_1'(\infty) = 0, \tag{29a)–(29d}$$

(b)

$$f_1(0) = \phi_1'(0) = \phi_1(\infty) = 0,$$

$$f_1'(\infty) = -\tfrac{2}{3}\cot\left(\frac{2\pi}{3}\right)f_0(\infty), \tag{30a)–(30d}$$

(c)

$$f_1(0) = \phi_1'(0) = \phi_1(\infty) = 0,$$

$$f_1'(\infty) = -\tfrac{1}{3}\cot\left(\frac{\pi}{3}\right)f_0(\infty), \tag{31a)–(31d}$$

where $f_1'(\infty)$ was obtained by matching with the first-order outer solution. Equations (27) and (28) are those also obtained in ref. [3]. Their boundary conditions for (a) are the same as here, since $(t_o - t_\infty) = (t_o - t_\infty)_0$ for the isothermal condition. However, for $n = 1/3$, the constant flux surface, $\phi_1(0) = 0$ has been taken. The correct condition for the uniform flux condition is $\phi_1'(0) = 0$. The plume, (c), has not been examined in ref. [3].

For the isothermal condition $f_1'(0) = \phi_1(0) = 0$, in view of the homogeneous boundary conditions. For (b) and (c) numerical integration yields up to four digits:

(b)

$$f_1'(0) = 0.1677,\quad \phi_1(0) = -0.3333, \tag{32}$$

(c)

$$f_1'(0) = 0,\qquad \phi_1(0) = 0.4714, \tag{33}$$

where

$$f_1(\eta) \to -\left(\frac{n+1}{2}\right)\eta f_0(\infty)\cot\left\{\frac{(n+1)\pi}{2}\right\} + A_1 \quad \text{as} \quad \eta\to\infty.$$

The second-order outer solution is governed by

$$\nabla^2\tilde{\psi}_2 = 0;\quad \tilde{\psi}_2(\theta = \pi) = 0,\quad \tilde{\psi}_2(\theta = 0) = A_1\alpha. \tag{34}$$

The condition at $\theta = 0$ results from matching a two term inner expansion with a two term outer expansion. The solution to equation (34) is given by

$$\tilde{\psi}_2 = A_1\alpha\left[1 - \frac{\theta}{\pi}\right]. \tag{35}$$

In the matching region

$$\tilde{\psi}_2 = A_1\alpha(Ra_x)^{1/2}\varepsilon\left[1 - \frac{\varepsilon\eta}{\pi} + \frac{\varepsilon^3\eta^3}{3\pi} + \text{h.o.t.}\right]. \tag{36}$$

The second-order inner problem is next obtained as

$$f_2''+\frac{(n-1)^2}{4}\eta^2 f_0''+\frac{(3n^2-4n+1)}{4}\eta f_0'+\frac{(n^2-1)}{4}f_0-\phi_2'=0, \quad (37)$$

$$\phi_2''+\frac{(n-1)^2}{4}\eta^2\phi_0''+\frac{(5n^2-8n+3)}{4}\eta\phi_0'+n(n-1)\phi_0+f_0'\phi_2-\frac{(n-1)}{2}f_1'\phi_1-nf_2'\phi_0-\frac{(n+1)}{2}f_2\phi_0'+\frac{(n+1)}{2}f_0\phi_2'=0, \quad (38)$$

with the boundary conditions:

for (a)

$$f_2(0)=\phi_2(0)=\phi_2(\infty)=0,\quad f_2''(\infty)=\frac{f_0(\infty)}{4}, \quad (39)$$

for (b)

$$f_2(0)=\phi_2'(0)=\phi_2(\infty)=0,\quad f_2''(\infty)=\tfrac{2}{9}f_0(\infty), \quad (40)$$

for (c)

$$f_2(0)=\phi_2'(0)=\phi_2(\infty)=0,\quad f_2''(\infty)=\tfrac{2}{9}f_0(\infty), \quad (41)$$

where $f_2'(\eta)\to f_0(\infty)\{[(1-n^2)/4]\eta\}-A_1/\pi$ as $\eta\to\infty$.

As discussed in the next section, for (a), the first eigenfunction appears at $O(\varepsilon^2)$ and hence equations (37) and (38) do not hold good. Numerical solution for the other two conditions yields:

for (b)

$$f_2'(0)=0.3442,\quad \phi_2(0)=0.0910,$$

for (c)

$$f_2'(0)=0.2823,\quad \phi_2(0)=0.4760.$$

The nonhomogeneous boundary condition in equations (39)–(41) results from matching the three term outer solution (i.e. $\tilde{\psi}_0+\tilde{\psi}_1+\tilde{\psi}_2$) with the three term inner solution.

The eigenvalues and eigenfunctions which arise are next discussed. To the inner expansions (7) and (8), eigenfunctions may be added which satisfy the boundary conditions at $\eta=0$ and $\eta=\infty$. Any combination of these eigenfunctions may be added to the inner solution. The multiplicative constants associated with these eigenfunctions are generally indeterminate.

The eigenfunctions associated with equations (7) and (8) are of the form $C_k\varepsilon^{\alpha_k}\alpha Ra_x^{1/2}F_k$ and $C_k\varepsilon^{\alpha_k}\,d\theta_k$, respectively. Here α_k is the kth eigenvalue and C_k is the multiplicative constant associated with it. Inclusion of these terms in the inner expansions and their substitution into equations (2) and (3) yields

$$F_k'-\theta_k=0, \quad (42)$$

$$\theta_k''-nF_k'\phi_0-\left\{n-\frac{\alpha_k(n+1)}{2}\right\}f_0'\theta_k+\frac{(n+1)}{2}f_0\theta_k'+\frac{(n+1)}{2}(1-\alpha_k)\phi_0'F_k=0, \quad (43)$$

with the boundary conditions:

for (a)

$$\theta_k(0)=F_k(0)=\theta_k(\infty)=0, \quad (44)$$

for (b) and (c)

$$\theta_k'(0)=F_k(0)=\theta_k(\infty)=0. \quad (45)$$

Nontrivial solutions to the above set of equations and boundary conditions exist only for discrete values of α_k.

In general the eigenvalues must be determined numerically. However, the first eigenvalue is normally associated with the shift in the location of the leading edge and the first eigenfunction is proportional to the x derivative of the zero-order expansions in equations (7) and (8). Stewartson [8] has found inconsistency in the behavior of the solution of boundary layer problems as $\eta\to\infty$, when a term in the postulated expansions is of the same order as one of the eigenfunctions. The inconsistency was resolved by inserting a log or log log term in the expansions.

The first eigenvalue in this problem is determined to be

$$\alpha_1=\frac{2}{(n+1)}. \quad (46)$$

For the isothermal condition, the first eigenvalue occurs at ε^2, which is of the same order as the second-order term in the expansions (7) and (8). Thus, logarithmic terms must be added to the expansions to get consistent solutions as $\eta\to\infty$. These terms are of the same order, that is $O(\varepsilon^2)$, the constant associated with them being indeterminate. The indeterminate constant can only be estimated approximately by using detailed finite difference techniques. This is beyond the scope of the present work. Thus the isothermal condition gives no higher-order correction up to $O(\varepsilon)$ and the $O(\varepsilon^2)$ correction is indeterminate.

For the constant flux condition, equation (46) yields $\alpha_1=3/2$. The first eigenfunction thus arises at $O(\varepsilon^{3/2})$. The eigenfunctions associated with $\alpha_1=3/2$ are given by

$$F_1=C[2f_0-\eta f_0'],\quad \theta_1=C_1[\phi_0-\eta\phi_0'], \quad (47)$$

where C_1 is a constant. The next two eigenvalues determined by a numerical 'shooting' method are [with the normalization condition $\theta_1(0)=1$]

$$\alpha_2=5.09,\quad \alpha_3=13.54.$$

Although the inclusion of the log term into equations (7) and (8) is not appropriate for the constant flux condition, arbitrary multiples of equations (47) can still be added to equations (7) and (8). However, it can be shown [see Appendix A] that at least $C_1\equiv 0$. Thus expansions (7) and (8) are accurate up to $O(\varepsilon^2)$. For the

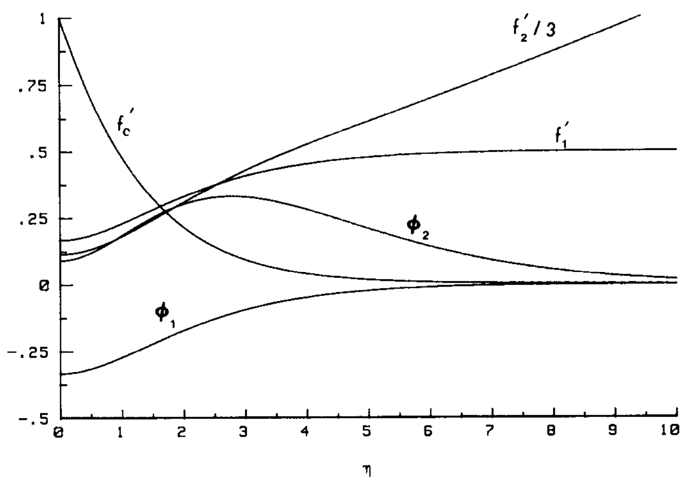

FIG. 1. The zero-order boundary layer solution f'_0, ϕ_0 and the correction functions f'_1, ϕ_1, f'_2, ϕ_2 for a constant flux surface in a fluid saturated porous medium.

plane plume, equation (46) yields $\alpha_1 = 3$. Thus, expansions (7) and (8) are appropriate for the line plume up to $O(\varepsilon^2)$.

RESULTS AND CONCLUSIONS

The zero-order solutions f'_0, ϕ_0 and the correction functions f'_1, f'_2, ϕ_1, ϕ_2 are plotted in Figs. 1 and 2 for both the constant flux surface and the plane plume. No results have been plotted for the isothermal surface, as the $O(\varepsilon)$ contribution identically vanishes and the $O(\varepsilon^2)$ correction becomes indeterminate.

Centerline temperature decay up to $O(\varepsilon^2)$ for the constant flux condition and the plane plume are obtained as follows:

for (b)

$$(t_o - t_\infty) = (t_o - t_\infty)_0[1 - 0.3333\varepsilon + 0.0910\varepsilon^2], \quad (48)$$

for (c)

$$(t_o - t_\infty) = (t_o - t_\infty)_0[1 + 0.4714\varepsilon + 0.4760\varepsilon^2] \quad (49)$$

where $(t_o - t_\infty)_0$ is:

for (b)

$$(t_o - t_\infty)_0 = [q''^2\mu\alpha/[k^2\{\phi'_0(0)\}^2\rho g\beta K]]^{1/3} \cdot x^{1/3} \quad (50)$$

for (c)

$$(t_o - t_\infty)_0 = \left[Q^2\mu \Big/ \left[24\rho^3 c_p^2 \alpha g\beta K \times \left\{\int_0^\infty \operatorname{sech}^4 \xi \, d\xi\right\}^2\right]\right]^{1/3} x^{-1/3}. \quad (51)$$

The local Nusselt number for the constant flux

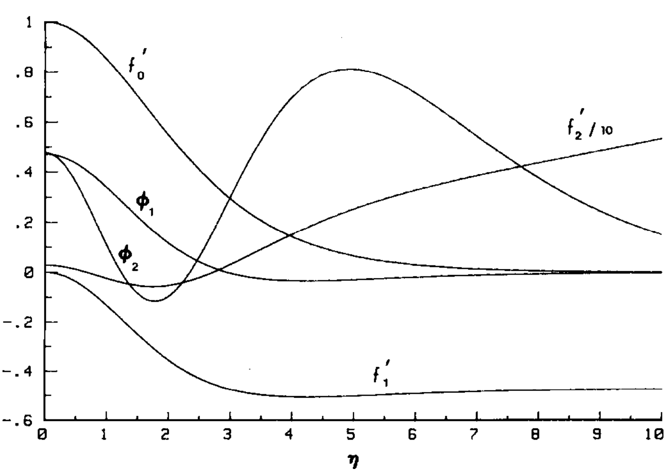

FIG. 2. The zero-order boundary layer solution f'_0, ϕ_0 and the correction functions f'_1, ϕ_1, f'_2, ϕ_2 for a line plume in a fluid saturated porous medium.

surface is given by

$$Nu_x \equiv \frac{hx}{k} = \frac{q''x}{k(t_o - t_\infty)}$$

$$= \frac{q''x}{k[1-0.3333\varepsilon+0.0910\varepsilon^2](t_o-t_\infty)_0}. \quad (52)$$

The ratio of the local Nu_x in equation (52) with the Nusselt number obtained from the zero-order boundary layer solution is given by

$$\frac{Nu_x}{(Nu_x)_{b.l.}} = \frac{(t_o - t_\infty)_0}{(t_o - t_\infty)} = 1+0.3333\varepsilon+0.0201\varepsilon^2. \quad (53)$$

The centerline velocity for the plume is given by

$$u(x,0) = \frac{\alpha}{x} Ra_x[1+0.2823\varepsilon^2]. \quad (54)$$

In many geothermal applications the maximum Rayleigh number is often of the order of 500. The results of the boundary layer similarity solution in ref. [1] were applied to an isothermal dike at 200°C in an aquifier at 15°C. The resulting Rayleigh number at $x = 300$ m is 575. These values are typical of actual conditions. Using $(t_o - t_\infty)_0 = 185$°C at $x = 10$ m and the same fluid properties as used in ref. [1], $Ra_x \cong 20$, resulting in $\varepsilon = 0.22$. This gives a 7% correction in equation (48) and a 13% correction in equation (49). The correction in the maximum centerline velocity of a plume, given in equation (54), is only about 1%. Thus for small downstream distances corrections to the simplest boundary layer temperature computations may indeed be appreciable.

Finally, we note that an analysis similar to that presented here is possible for the horizontal flows, to extend the results of Cheng and Chang [3].

Acknowledgements—We acknowledge communications with Dr C. A. Hieber of Cornell University on the numerical calculations of eigenvalues. This work was carried out under Grant N.S.F. MEA8200613.

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APPENDIX A

The integrated form of the energy equation (3) across the boundary layer, for the constant flux surface is given by

$$\rho c_p \int_0^\infty u(t-t_\infty)\,dy = k\int_0^\infty \frac{\partial t}{\partial x}\,dy + q''x. \quad (A1)$$

Expansions (7) and (8) are substituted in equation (A1) and equal powers of ε collected, noting that

$$q'' = -k\left.\frac{\partial t}{\partial y}\right|_{y=0} = -kd(x)\frac{Ra_x^{1/2}}{x}\phi_0'(0).$$

The resulting equations up to $O(\varepsilon^{3/2})$ are

$$\varepsilon^0: \quad \int_0^\infty (f_0'\phi_0) = -\phi_0'(0), \quad (A2)$$

$$\varepsilon^1: \quad \int_0^\infty (f_0'\phi_1 + f_1'\phi_0)\,d\eta = 0, \quad (A3)$$

$$\varepsilon^{3/2}: \quad C_1\int_0^\infty (F_1'\phi_0 + f_0'\theta_1)\,d\eta = 0. \quad (A4)$$

Equations (A2) and (A3) are identical to the integrated forms of equations (13) and (28), respectively. F_1 and θ_1 are given by equation (47). Substitution of these into equation (A4) yields

$$C_1\int_0^\infty (f_0' - \eta\phi_0')\phi_0\,d\eta = 0. \quad (A5)$$

By inspecting Fig. 1 it is clear that $f_0' > 0$ and $\phi_0' < 0$ for all $0 \leqslant \eta < \infty$. The only way equation (A5) can then be satisfied is when $C_1 \equiv 0$.

ECOULEMENTS VERTICAUX DE CONVECTION NATURELLE DANS LES MILIEUX POREUX: CALCULS AVEC PRECISION ACCRUE

Résumé—Des corrections d'ordre élevé dans les solutions de couche limite sont obtenues pour les écoulements verticaux de convection naturelle dans les milieux poreux. L'analyse de Cheng et Chang [*Lett. Heat Mass Transfer* **6**, 253–258 (1979)] est étendue à une condition de flux thermique uniforme sur la paroi et à des écoulements de panache plan. Des solutions asymptotiques jusqu'à $O(\varepsilon^2)$ sont obtenues pour ces variations de température. Pour la condition de surface isotherme, il n'y a pas de correction ni pour la température, ni pour la vitesse jusqu'à $O(\varepsilon^2)$. On trouve néanmoins que la première fonction propre pour cette condition coïncide avec le terme $O(\varepsilon^2)$ dans le développement interne. Ceci rend indéterminée la correction du second ordre. Pour l'écoulement à flux pariétal uniforme, le rapport du nombre de Nusselt local corrigé à sa valeur pour le plus simple résultat de couche limite est

$$1+0{,}3333\varepsilon+0{,}0201\varepsilon^2.$$

Pour le panache plan, le rapport de l'excès de température sur la ligne des centres à sa valeur pour la solution simple est

$$1+0{,}4714\varepsilon+0{,}4760\varepsilon^2.$$

Le nombre de Prandtl est absorbé dans les transformations. Les nouveaux résultats sur les quantités de transport sont plus précis pour ces écoulements dans le domaine de nombre de Rayleigh d'intérêt pratique.

SENKRECHTE NATÜRLICHE KONVEKTION IN PORÖSEN MEDIEN: BERECHNUNGEN VON ERHÖHTER GENAUIGKEIT

Zusammenfassung—Es ergeben sich Verbesserungen von höherer Ordnung bei den Grenzschicht-Lösungen für die senkrechten natürlichen Konvektionsströmungen in porösen Medien. Das Berechnungsverfahren von Cheng und Chang [*Lett. Heat Mass Transfer* **6**, 253–258 (1979)] wurde für gleichmäßige Wärmestromdichte in der Oberfläche und für ebene Strömungsfahnen erweitert. Passende asymptotische Lösungen bis zu $O(\varepsilon^2)$ ergaben sich für die stromabwärts veränderlichen Temperaturen. Für die Bedingung der isothermen Oberfläche war weder für die Temperatur noch für die Geschwindigkeit eine Verbesserung bis zu $O(\varepsilon)$ möglich. Jedoch wurde herausgefunden, daß die erste Eigenfunktion für diese Bedingung mit dem $O(\varepsilon^2)$-Ausdruck in der inneren Entwicklung übereinstimmt. Dadurch wird die Verbesserung zweiter Ordnung unbestimmbar. Für die Oberfläche mit gleichförmigem Wärmestrom ist das Verhältnis aus korrigierter örtlicher *Nu*-Zahl und ihrem Wert nach der einfachsten Grenzschichtlösung

$$1+0{,}3333\varepsilon+0{,}0201\varepsilon^2.$$

Für die ebene Strömungsfahne ist das Verhältnis aus den in der Mittellinie stromab vorhandenen Übertemperatur und ihrem Wert nach der einfachsten Lösung

$$1+0{,}4714\varepsilon+0{,}4760\varepsilon^2.$$

Die Prandtl-Zahl wird in den Umformungen eliminiert. Die neuen Ergebnisse liefern für den Transport in diesen Strömungen im praktisch interessierenden Bereich der Rayleigh-Zahl genauere Werte.

ВЕРТИКАЛЬНЫЕ ЕСТЕСТВЕННЫЕ КОНВЕКТИВНЫЕ ТЕЧЕНИЯ В ПОРИСТЫХ СРЕДАХ: РАСЧЕТЫ ПОВЫШЕННОЙ ТОЧНОСТИ

Аннотация—Поправки более высокого порядка были получены к решениям уравнений пограничного слоя для случая вертикальных естественных конвективных течений в пористых средах. Анализ Ченга и Чанга [3] был применен для случая однородного теплового потока на поверхности и для плоских всплывающих потоков. Были получены сращиваемые асимптотические решения вплоть до $O(\varepsilon^2)$ для заданных изменений температуры по потоку. В случае изотермической поверхности отсутствовали поправки как для температуры, так и для скорости вплоть до $O(\varepsilon)$. Однако было установлено, что первая собственная функция для такого случая совпадает с членом внутреннего разложения $O(\varepsilon^2)$. Это приводит к неопределенности поправки второго порядка. Для течения при неоднородном по поверхности потоке отношение уточненного местного числа Нуссельта к его значению для простейшего решения пограничного слоя составляет

$$1+0{,}3333\varepsilon+0{,}0201\varepsilon^2.$$

Для плоского всплывающего потока отношение избытка температуры вдоль осевой линии потока к его значению в случае простейшего решения составляет

$$1+0{,}4714\varepsilon+0{,}4760\varepsilon^2.$$

Число Прандтля сокращается в процессе преобразований. Новые результаты являются более точными для значений коэффициентов переноса в диапазоне чисел Рэлея, представляющих практический интерес.